Int. J. Solids Structures, 1968, Vol. 4, pp. 1181 to 1201. Pergamon Press. Printed in Great Britain

STRESS DIFFUSION FROM AXIALLY LOADED STIFFENERS INTO CYLINDRICAL SHELLS*

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Abstract—A load diffusion problem for cylindrical elastic shells is solved. Infinitely long shells of circular cross section and uniform thickness are considered. These cylinders are reinforced by one or more equally spaced, continuously attached axial stiffeners. Each stiffener is loaded by a single concentrated longitudinal force. The magnitude, sense and axial location of application of each force are the same. The stress quantities of interest are the membrane shearing stress transmitted by a loaded stringer to the shell and the axial stress developed within the stringer. The extent to which shell curvature, stringer spacing and stringer bending stiffness affect these stress quantities is illustrated.

NOTATION

A_c area of cylinder cross section, $2\pi Rh$
A_s area of stringer cross section
$\bar{A}$ dimensionless area ratio, $2\pi E_s A_s / E A_c$
B^2 dimensionless bending stiffness parameter, $6(1-\nu^2)I_s/A_s h^2$
C_ν $(3-\nu)(1+\nu)/4$
D shell bending stiffness parameter, $Eh^3/12(1-\nu^2)$
E elastic modulus of shell
E_s elastic modulus of stringer
F stress potential function for Simmonds' theory
$\bar{F}$ $(Eh/PE_sA_s)F$
h shell thickness
$H(x)$ Heaviside step function
I_s area moment of inertia of stringer
$M_{\alpha\beta}$ shell bending moment, ($\alpha, \beta = 1$ or 2)
$N_{\alpha\beta}$ shell membrane stress, $\phi_{,\rho\rho}\delta_{\alpha\beta} - \phi_{,\alpha\beta}$.
$\bar{N}_{\alpha\beta}$ $(E_sA_s/PEh)N_{\alpha\beta}$
N_s number of stringers
P magnitude of applied concentrated force
R shell radius of curvature
u axial displacement component
v circumferential displacement component
w displacement component normal to shell surface
$\bar{w}$ $(E_sA_s/PR)w$
x physical axial coordinate measure
y physical circumferential coordinate measure
β^4 $12(1-\nu^2)\gamma^4$
γ dimensionless curvature parameter, $E_sA_s/E(Rh^3)^{\frac{1}{2}}$
$\delta(\xi)$ Dirac delta function
$\delta_{\alpha\beta}$ Kronecker delta
$\bar{\nabla}^4$ biharmonic operator in (ξ, η)
$\varepsilon_{\alpha\beta}$ membrane strain component
ε_s axial strain of stringer

* This work was supported in part by the National Aeronautics and Space Administration under Grant NsG-559, and by the Division of Engineering and Applied Physics, Harvard University.
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ζ $(1/C_v)\xi$
η dimensionless circumferential coordinate measure, $(Eh/E_sA_s)y$
$\bar{\eta}$ $\pi/\bar{A}N_s$
ν Poisson's ratio
λ_0 $\bar{A}^2/12(1-\nu^2)\gamma^4$
ξ dimensionless axial coordinate measure, $(Eh/E_sA_s)x$
σ_s physical stringer stress
$\bar{\sigma}$ $(A_s/P)\sigma_s$
$\bar{\tau}$ dimensionless shear stress measure, $2C_v\bar{N}_{xy}$
ϕ stress potential function for shallow shell theory
$\bar{\Phi}$ $(Eh/PE_sA_s)\phi$
ω transform variable

INTRODUCTION AND STATEMENT OF THE PROBLEM

THE problem of load diffusion from a stiffener into a thin elastic sheet is one of significant technical importance. It is encountered in many areas of structural analysis and has particular relevance to the design of aeronautical structures. The earliest work in this area appears to be the classical paper of Melan [1] in which the transfer of an axial concentrated load from an infinite stiffener to an infinite flat sheet is examined. Solutions to additional planar problems have subsequently been obtained by Koiter [2], Muki and Sternberg [3] and others*. In each of these analyses the flat sheet was treated exactly within the context of generalized plane stress, although various degrees of approximation were used in modeling the stiffener.

Approximate theories have also been suggested and employed for the analysis of flat sheet–stringer problems. In some of the more familiar "shear lag" theories analytical simplification is obtained through the use of approximate constitutive relations for the flat sheet.

With regard to the treatment of load diffusion for axially stiffened cylindrical shells, there appear to be no "exact" solutions within the context of shell theory. Several approximate analyses appear in the literature such as the semi-empirical shear lag approach of Kuhn [4] and the more sophisticated work of McComb [5]. McComb's analysis for circular cylindrical shells stiffened by both stringers and rings is based on the following assumptions regarding the properties of the structure:

(a) The stringers carry only direct stress and the sheet takes only shear stress which is constant within each shear panel; thus stringer stresses vary linearly between adjacent rings.
(b) The rings are uniform and have a finite bending stiffness in their own planes, but they do not restrain longitudinal displacements of the stringers. The bending of the rings is inextensional.
(c) Effective stringer and ring properties are a composite of actual stiffener properties and contributions due to the discretization of certain shell properties.

In the present work a number of problems are treated exactly within the context of shell theory. The structure to be considered is an infinitely long circular cylindrical shell. This cylinder is reinforced along its entire length by one or more equally spaced, continuously attached axial stiffeners as shown in Fig. 1. The neutral axis of each stringer lies in the shell middle surface. A single concentrated axial force is applied to each stringer.

* See [3] for a more extensive bibliography of load diffusion problems for the flat sheet.

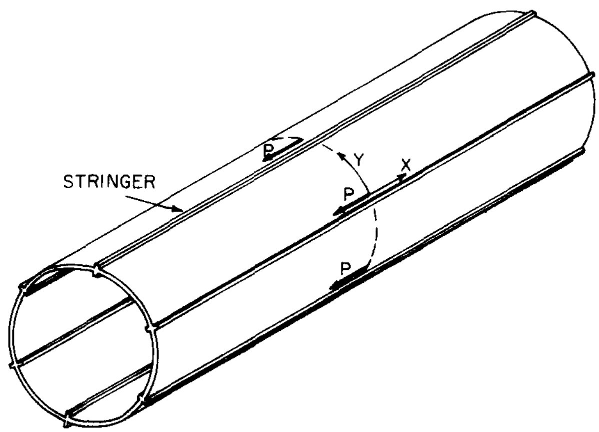

FIG. 1. Axially stiffened circular cylindrical shell with a single concentrated longitudinal force applied to each stringer.

These loads have the same magnitude, sense and axial location of application. The membrane shearing stress transmitted by a loaded stringer to the shell and the axial stress developed within the stringer will be sought.

A first approach to this problem will be through the use of the shallow shell equations of Marguerre [6]. In this treatment a cylindrical shell of parabolic cross section stiffened axially along its apex generator is analyzed. To assess the results of the shallow shell analysis the problem as originally described will be treated exactly with a set of shell equations due to Simmonds [7].

The results of the above analyses, the behavior of the transmitted shear and stringer normal stress, will be compared and contrasted with the flat sheet–stringer results of the Melan problem. The extent to which shell curvature, stringer spacing and stringer bending stiffness affect the process of load diffusion will be discussed.

SHALLOW SHELL APPROACH

Load diffusion from a stringer into a cylindrical shell is felt to be largely confined to and affected by a local neighborhood of the shell in the immediate vicinity of the loaded stringer. For this reason an approach based on the shallow shell equations of Marguerre [6] will be investigated. These equations will be used to treat an infinitely long cylindrical shell of parabolic cross section. This shell is stiffened axially along its apex generator and the stiffener is loaded by a single concentrated force directed along the stringer as shown in Fig. 2. The radius of curvature of the parabola at its apex is R.

Strictly speaking, the field equations of the Marguerre theory apply only to a shallow portion of the parabolic cylinder in which the rise-to-span ratio does not exceed about 1-to-8. The domain of application will be formally extended to the full parabolic shell, however, on the basis of the expected exponential decay of stress and displacement quantities in the circumferential direction; this behavior is anticipated on the basis of known problem solutions, e.g. Van Dyke [8].

Provided that stress and displacement quantities do decay to negligible values within a shallow portion of the shell containing the loaded stringer, the results of this single-

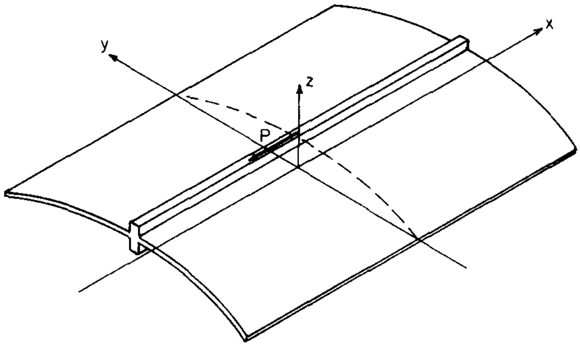

FIG. 2. Geometry of shallow cylindrical shell. Axial stringer loaded by a concentrated longitudinal force.

stringer analysis should be quite good for the description of the multi-stringer circular cylinder when adjacent stringers do not lie in a common shallow portion of the shell.

Boundary value problem

For the shell geometry under consideration, the field equations of the linear Marguerre theory formulated in terms of an Airy stress potential function ϕ, and the shell normal displacement component w, become

$$D\nabla^4 w+\frac{1}{R}\phi_{,xx} = 0 \tag{1}$$

$$\nabla^4\phi-\frac{Eh}{R}w_{,xx} = 0. \tag{2}$$

These equations are supposed applicable in the semi-infinite domain $(-\infty < x < \infty,$ $0 < y < \infty)$, i.e. in that portion of the shell to one side of the loaded stringer. The first of these equations is the expression of equilibrium of the shell in the direction normal to the shell surface, while the second is the statement of compatibility of strains.

The boundary conditions (hereafter also referred to as b.c.) to be satisfied along the line of stringer attachment $(-\infty < x < \infty, y = 0)$, will now be developed into expressions involving only ϕ and w. Symmetry of the shell deformation provides the basis for two of the four b.c. The first of these states that the tangential shell displacement normal to the stringer is zero along this boundary. More precisely,

$$v(x, y)|_{y=0} = 0. \tag{3}$$

The strain–displacement relations

$$\begin{aligned} \varepsilon_{xy,x} &= \tfrac{1}{2}(u_{,yx}+v_{,xx}) \\ \varepsilon_{xx,y} &= u_{,xy} \end{aligned} \tag{4}$$

used in conjunction with (3) yield the expression

$$\varepsilon_{xx,y}-2\varepsilon_{xy,x}|_{y=0} = 0. \tag{5}$$

Upon substitution of the constitutive relations

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{Eh}(N_{xx} - \nu N_{yy}) = \frac{1}{Eh}(\phi_{,yy} - \nu \phi_{,xx}) \\ \varepsilon_{xy} &= \frac{1+\nu}{Eh} N_{xy} = -\frac{1+\nu}{Eh}\phi_{,xy} \end{aligned} \tag{6}$$

where $N_{\alpha\beta} = \phi_{,\rho\rho}\delta_{\alpha\beta} - \phi_{,\alpha\beta}$, into (5) the final form for this b.c. is obtained as

$$\phi_{,yyy} + (2+\nu)\phi_{,xxy}|_{y=0} = 0. \tag{7}$$

The second b.c. states that the rotation of the shell in the circumferential direction along the line of stringer attachment must, by symmetry, vanish; thus

$$w_{,y}|_{y=0} = 0. \tag{8}$$

Another boundary condition is a statement of the axial force equilibrium of the stringer and can be written as

$$2\int_{-\infty}^{x} N_{xy}(\bar{x}, 0^+)\,\mathrm{d}\bar{x} + \sigma_s(x) \,.\, A_s - P \,.\, H(x) = 0. \tag{9}$$

The constitutive relation for the stringer is taken to be

$$\sigma_s(x) = E_s \varepsilon_s(x) \tag{10}$$

which, together with the condition of axial strain continuity between the stringer and shell, becomes

$$\sigma_s(x) = \frac{E_s}{Eh}(N_{xx} - \nu N_{yy})|_{y=0}. \tag{11}$$

Upon the introduction of (11) into (9) and the subsequent differentiation with respect to x, the following expression is obtained:

$$2N_{xy} + \frac{E_s A_s}{Eh}(N_{xx,x} - \nu N_{yy,x})|_{y=0} = P \,.\, \delta(x). \tag{12}$$

The final form for this b.c. is obtained by introduction of the stress potential function as before. Thus, (12) becomes

$$-2\phi_{,xy} + \frac{E_s A_s}{Eh}(\phi_{,xyy} - \nu \phi_{,xxx})|_{y=0} = P \,.\, \delta(x). \tag{13}$$

The fourth and final b.c. is a statement of the moment equilibrium of the stringer. This condition becomes

$$\tfrac{1}{2} E_s I_s w_{,xxxx} + D[(2-\nu) w_{,xxy} + w_{,yyy}]|_{y=0} = 0 \tag{14}$$

when elementary beam theory is used to describe the stringer. Equation (8) can be used to simplify (14) to

$$\tfrac{1}{2} E_s I_s w_{,xxxx} + D w_{,yyy}|_{y=0} = 0. \tag{15}$$

The field equations (1) and (2) together with the b.c. as expressed by (7), (8), (13) and (15) and the requirement of boundedness of unknowns at infinity constitute the boundary value problem for the semi-infinite domain ($-\infty < x < \infty$, $0 < y < \infty$) to be solved. Upon introduction of the non-dimensionalized quantities

$$\begin{aligned} \xi &= \frac{Eh}{E_s A_s} x \\ \eta &= \frac{Eh}{E_s A_s} y \\ \bar{\phi} &= \frac{Eh}{P E_s A_s} \phi \\ \bar{w} &= \frac{E_s A_s}{PR} w \end{aligned} \tag{16}$$

the field equations (1) and (2) become

$$\bar{\nabla}^4 \bar{w} + \beta^4 \bar{\phi}_{,\xi\xi} = 0 \tag{17}$$

$$\bar{\nabla}^4 \bar{\phi} - \bar{w}_{,\xi\xi} = 0 \tag{18}$$

respectively, where

$$\beta = [12(1-\nu^2)]^{\frac{1}{4}} \frac{E_s A_s}{E(Rh^3)^{\frac{1}{2}}} = [12(1-\nu^2)]^{\frac{1}{4}} \gamma \tag{19}$$

is a parameter providing a measure of the shell curvature. In a similar manner (7), (8), (13) and (15) become

$$\begin{aligned} \bar{\phi}_{,\eta\eta\eta} + (2+\nu)\bar{\phi}_{,\xi\xi\eta}|_{\eta=0} &= 0 \\ \bar{w}_{,\eta}|_{\eta=0} &= 0 \\ -2\bar{\phi}_{,\xi\eta} + \bar{\phi}_{,\xi\eta\eta} - \nu\bar{\phi}_{,\xi\xi\xi}|_{\eta=0} &= \delta(\xi) \\ B^2 \bar{w}_{,\xi\xi\xi\xi} + \bar{w}_{,\eta\eta\eta}|_{\eta=0} &= 0 \end{aligned} \tag{20}$$

respectively, where

$$B^2 = 6(1-\nu^2)\frac{I_s}{A_s h^2} \tag{21}$$

provides a measure of the stringer bending stiffness.

Solution and results

Fourier transforms will be used to effect the solution of the boundary value problem as stated in the previous section. The notation

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(\xi) \exp(i\omega\xi) \, d\xi \tag{22}$$

is adopted for the Fourier transform $\tilde{f}(\omega)$ of $f(\xi)$. The associated inversion formula is

$$f(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\exp(-i\xi\omega)\,d\omega. \tag{23}$$

Application of the Fourier transform in ξ to the field equations (17) and (18) yields, respectively

$$\tilde{w}_{,\eta\eta\eta\eta} - 2\omega^2\tilde{w}_{,\eta\eta} + \omega^4\tilde{w} - \beta^4\omega^2\tilde{\phi} = 0 \tag{24}$$

$$\tilde{\phi}_{,\eta\eta\eta\eta} - 2\omega^2\tilde{\phi}_{,\eta\eta} + \omega^4\tilde{\phi} + \omega^2\tilde{w} = 0 \tag{25}$$

for $-\infty < \omega < \infty$ and $0 < \eta < \infty$. Solutions of these equations for $\tilde{\phi}$ and $\tilde{w}$ are

$$\begin{aligned} \tilde{\phi}(\omega, \eta) &= A(\omega)\exp(\alpha\eta) \\ \tilde{w}(\omega, \eta) &= B(\omega)\exp(\alpha\eta) \end{aligned} \tag{26}$$

where $\alpha = \alpha(\omega)$ is determined from the characteristic equation

$$(\alpha^2 - \omega^2)^4 + \beta^4\omega^4 = 0. \tag{27}$$

The four values of α^2 satisfying (27) are

$$\alpha^2 = \omega^2 + \beta|\omega|\exp(in\pi/4), \qquad (n = 1, 3, 5, 7). \tag{28}$$

Due to the requirement of boundedness of solutions as $\eta \to \infty$, only those roots for $\alpha(\omega)$ satisfying (28) and having negative real parts will be retained. The four roots satisfying this condition are

$$\begin{aligned} \alpha_1(\omega) &= -|\omega|^{\frac{1}{2}}[(|\omega| + \rho) + i\rho]^{\frac{1}{2}} \\ \alpha_2(\omega) &= -|\omega|^{\frac{1}{2}}[(|\omega| - \rho) + i\rho]^{\frac{1}{2}} \\ \alpha_3(\omega) &= -|\omega|^{\frac{1}{2}}[(|\omega| - \rho) - i\rho]^{\frac{1}{2}} = \alpha_2^*(\omega) \\ \alpha_4(\omega) &= -|\omega|^{\frac{1}{2}}[(|\omega| + \rho) - i\rho]^{\frac{1}{2}} = \alpha_1^*(\omega) \end{aligned} \tag{29}$$

where $\rho = \beta/\sqrt{2} > 0$. The square roots occurring in (29) have the interpretation

$$z^{\frac{1}{2}} = +|z|^{\frac{1}{2}}\exp(i\theta/2) \tag{30}$$

where $z = |z|\exp(i\theta)$, $|z|^{\frac{1}{2}} > 0$ and θ takes on its principal value, i.e. $-\pi < \theta < \pi$.

The complete solutions for $\tilde{\phi}$ and $\tilde{w}$ with as yet unknown coefficients can be written as

$$\begin{aligned} \tilde{\phi} &= \sum_{n=1}^{4} A_n(\omega)\exp(\alpha_n\eta) \\ \tilde{w} &= \sum_{n=1}^{4} B_n(\omega)\exp(\alpha_n\eta) \end{aligned} \tag{31}$$

where the A_n's and B_n's are related according to the expression

$$B_n = (-1)^n i\beta^2 A_n, \qquad (n = 1, 2, 3, 4) \tag{32}$$

The four independent coefficients appearing in the solutions (31) are determined by enforcing the four b.c. given by (20). The appropriate form of these equations is obtained by application of the Fourier transform in ξ as before. One thus obtains

$$\begin{aligned} \tilde{\Phi}_{,\eta\eta\eta} - (2+\nu)\omega^2 \tilde{\Phi}_{,\eta}|_{\eta=0} &= 0 \\ \tilde{w}_{,\eta}|_{\eta=0} &= 0 \\ -2\tilde{\Phi}_{,\eta} + \tilde{\Phi}_{,\eta\eta} + \nu\omega^2\tilde{\Phi}|_{\eta=0} &= i/\omega \\ B^2\omega^4\tilde{w} + \tilde{w}_{,\eta\eta\eta}|_{\eta=0} &= 0 \end{aligned} \tag{33}$$

for these conditions.

Recalling that

$$\bar{N}_{xy} = -\bar{\Phi}_{,\xi\eta} \tag{34}$$

and hence, upon transformation in ξ,

$$\tilde{N}_{xy}|_{\eta=0} = i\omega\tilde{\Phi}_{,\eta}|_{\eta=0} = i\omega \sum_{n=1}^{4} A_n(\omega)\alpha_n(\omega) \tag{35}$$

we shall focus attention on the quantity

$$S(\omega) \equiv \sum_{n=1}^{4} A_n(\omega)\alpha_n(\omega). \tag{36}$$

A straightforward if somewhat tedious application of the b.c. (33) in conjunction with solutions (31) and the definition for $S(\omega)$ given by (36) yields, by a process of matrix inversion, an explicit formula for $S(\omega)$ in terms of the quantities $\alpha_n(\omega)$. For the general case of arbitrary bending stiffness of the stringer, i.e. $B^2 \geq 0$,

$$S(\omega) = -2iC\,.\,(F+G-H+J)^{-1} \tag{37}$$

where

$$\begin{aligned} C &= \mathrm{Re}[(m^* - n^*)(b-a)] \\ F &= 2\,.\,\mathrm{Re}[f\,.\,m^*]\,.\,\mathrm{Re}[b-d] \\ G &= 2\,.\,\mathrm{Re}[g\,.\,n^*]\,.\,\mathrm{Re}[a-d] \\ H &= \mathrm{Re}[(f\,.\,n^* + g^*\,.\,m)(b+a^*-2d)] \\ J &= \mathrm{Re}[(g^*\,.\,m^* - f^*\,.\,n^*)(b-a)] \end{aligned} \tag{38}$$

and

$$\begin{aligned} a &= \alpha_1^{\,2} \\ a^* &= \text{complex conjugate of } a \\ b &= \alpha_3^{\,2} \\ d &= (2+\nu)\omega^2 \\ f &= (2\alpha_1 - \alpha_1^{\,2} - \nu\omega^2)\omega/\alpha_1 \\ g &= (2\alpha_3 - \alpha_3^{\,2} - \nu\omega^2)\omega/\alpha_3 \\ m &= (\alpha_1^{\,3} + B^2\omega^4)/\alpha_1 \\ n &= (\alpha_3^{\,3} + B^2\omega^4)/\alpha_3. \end{aligned} \tag{39}$$

When $B^2 = 0$, corresponding to zero bending stiffness of the stringer, (37) assumes the more simple form

$$S(\omega) = i\,.\,[\omega\,.\,\mathrm{Re}(A)]^{-1} \tag{40}$$

where

$$\begin{aligned} A = \{&(\alpha_1{}^2-2\alpha_1+\omega^2\nu)[\alpha_2^{*2}-(2+\nu)\omega^2]\alpha_2^* \\ &-(\alpha_2^{*2}-2\alpha_2^*+\omega^2\nu)[\alpha_1{}^2-(2+\nu)\omega^2]\alpha_1\}. \\ &\{\alpha_1\,.\,\alpha_2^*(\alpha_2^{*2}-\alpha_1{}^2)\}^{-1}. \end{aligned} \tag{41}$$

Note that even for $B^2 = 0$, $S(\omega)$ is an extremely complicated algebraic expression containing many nested irrational functions of ω.

For purposes of comparison with the solutions for the Melan problem the following measure of shearing stress will be employed:

$$\bar{\tau}(\zeta) = 2C_\nu \bar{N}_{xy}(\xi, \eta)|_{\eta=0} \tag{42}$$

Here $\xi = C_\nu\zeta$ and $C_\nu = \frac{1}{4}(3-\nu)(1+\nu)$. This measure of shearing stress resulted from the canonical non-dimensionalization of the Melan problem. An integral representation for $\bar{\tau}(\zeta)$ obtained with the aid of inversion formula (23) is

$$\bar{\tau}(\zeta) = \frac{1}{\pi}\int_0^\infty 2\tilde{N}_{xy}\left(\frac{s}{C_\nu}, \eta\right)\Big|_{\eta=0} \cos(s\zeta)\,\mathrm{d}s, \qquad 0 < \zeta < \infty \tag{43}$$

where the evenness of $\tilde{N}_{xy}$ in ω has been used and $s = C_\nu\omega$.

A similar integral expression can likewise be obtained for the axial stress in the loaded stringer. Use of the equation for axial equilibrium of the stringer yields the relation

$$\tilde{\sigma}(\omega) = \frac{i}{\omega}[1-2\tilde{N}_{xy}(\omega, \eta)|_{\eta=0}]. \tag{44}$$

The inversion formula (23) applied to (44) gives

$$\bar{\sigma}(\zeta) = \frac{1}{\pi}\int_0^\infty \left\{\frac{1}{s}\left[1-2\tilde{N}_{xy}\left(\frac{s}{C_\nu}, \eta\right)\Big|_{\eta=0}\right]\right\}\sin(s\zeta)\,\mathrm{d}s, \qquad 0 < \zeta < \infty \tag{45}$$

where the oddness of $\tilde{\sigma}(\omega)$ has been used and ζ was introduced as before. The above solutions, i.e. (43) and (45), can be written succinctly as

$$\begin{aligned} \bar{\tau}(\zeta) &= \frac{1}{\pi}\int_0^\infty \tilde{\tau}(s)\cos(s\zeta)\,\mathrm{d}s \\ \bar{\sigma}(\zeta) &= \frac{1}{\pi}\int_0^\infty \tilde{\sigma}(s)\sin(s\zeta)\,\mathrm{d}s \end{aligned} \tag{46}$$

where $0 < \zeta < \infty$; it should be noted that, for $s > 0$,

$$\tilde{\sigma}(s) = \frac{1}{s}[1-\tilde{\tau}(s)]. \tag{47}$$

The integrals of (46) were evaluated numerically for small to moderately large values of ζ and by asymptotic techniques for large values of ζ tending to infinity. It was extremely helpful to note that both $\bar{\tau}(s)$ and $\bar{\sigma}(s)$ asymptote rapidly to the function $1/(1+s)$ for large, positive s. This simple function is the transform function, for $s > 0$, for both the shear stress and the axial stringer stress encountered in Melan's problem. For this reason, and also because both the cosine and sine transforms of $1/(1+s)$ are known, attention is directed to the numerical quadrature of the difference functions

$$\begin{aligned} \tilde{f}(s) &= 1/(1+s)-\bar{\tau}(s) \\ \tilde{g}(s) &= 1/(1+s)-\bar{\sigma}(s). \end{aligned} \tag{48}$$

The integrals thus treated by numerical quadrature were

$$\begin{aligned} &\int_0^{L_1} \tilde{f}(s)\cos(s\zeta)\,\mathrm{d}s \\ &\int_0^{L_2} \tilde{g}(s)\sin(s\zeta)\,\mathrm{d}s \end{aligned} \tag{49}$$

where the upper limits L_1 and L_2 for these integrals are large enough so that integration above these limits does not make any observable contribution to the integrals as a whole.

The problem of convergence due to the rapid oscillation of the integrand which occurs for large values of ζ is greatly reduced by use of a quadrature technique first introduced by Filon [9]. This technique approximates the enveloping part of the integrand by a quadratic function and then provides for the analytical integration of this quadratic estimate and the rapidly oscillating function over small intervals. This method is superior to Simpson's quadrature approach, for example, which approximates the entire integrand by a quadratic function and hence requires much finer increments for comparable accuracy. As a check on the performance of this technique for large values of ζ, the asymptotic behavior for both $\bar{\tau}(\zeta)$ and $\bar{\sigma}(\zeta)$ was found from formal asymptotic expansions generated through the use of Watson's Lemma.

Two elastic-geometric parameters have been noted to affect the solutions for both $\bar{\tau}(\zeta)$ and $\bar{\sigma}(\zeta)$. These parameters are $\gamma = E_s A_s/E(Rh^3)^{\frac{1}{2}}$, a measure of the shell curvature, and $B^2 = 6(1-\nu^2)I_s/(A_s h^2)$, a measure of the stringer bending stiffness. For representative values of γ the dependence of the solutions upon B^2 was investigated. The limiting extremes for B^2 are zero and infinity which correspond, respectively, to zero and infinite resistance to bending. A paper by Hutchinson and Amazigo [10] was helpful as a guide to realistic finite values of B^2. The following table is given for the purpose of nominal stiffness classification:

Stiffness	B^2
None	0
Light	25
Medium	50
Heavy	250
Infinite	∞

For small shell curvature as reflected by small γ, say $\gamma \leq 1$, the difference between the limiting solutions for $B^2 = 0$ and $B^2 = \infty$ was found to be quite small. This result is illustrated by Fig. 3* showing $\bar{\sigma}(\zeta)$ vs. ζ for the case of $\gamma = 1$. This result is somewhat to be

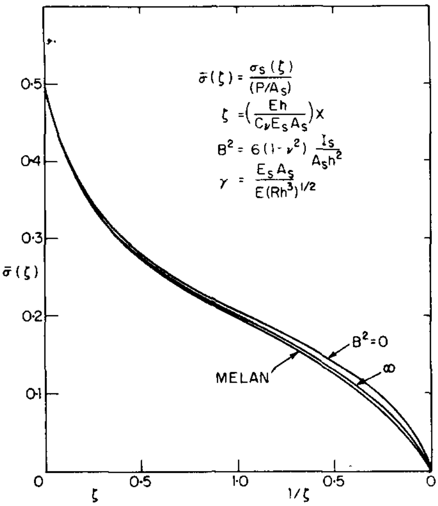

FIG. 3. $\bar{\sigma}(\zeta)$ vs. ζ, shallow shell theory; $\gamma = 1$.

expected since in the case of zero curvature ($\gamma = 0$) $w \equiv 0$ and bending stiffness of the stringer is completely irrelevant. It is also noticed that the solutions for $\gamma = 1$ do not differ substantially from the results of the Melan problem. As the shell curvature is allowed to increase, however, the enveloping solutions for $B^2 = 0$ and $B^2 = \infty$ become farther apart as well as quite different from the Melan results. The behavior of $\bar{\sigma}(\zeta)$ vs. ζ for $\gamma = 5$ and $\gamma = 10$ is shown in Figs. 4 and 5 respectively, for $B^2 = 0, 50, 250$ and ∞. These figures give

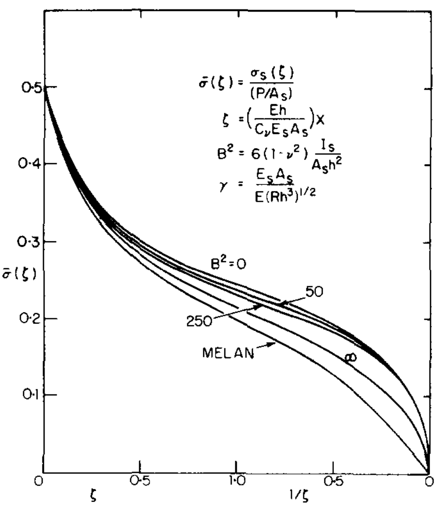

FIG. 4. $\bar{\sigma}(\zeta)$ vs. ζ, shallow shell theory; $\gamma = 5$.

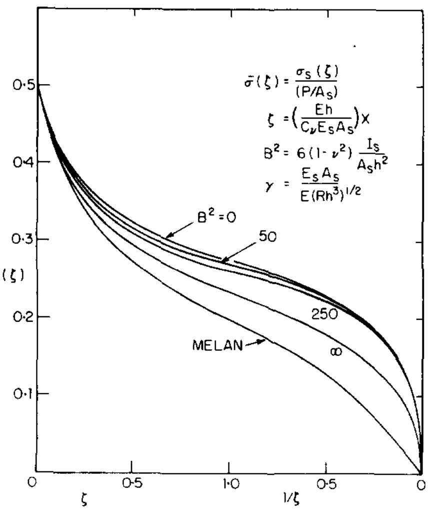

FIG. 5. $\bar{\sigma}(\zeta)$ vs. ζ, shallow shell theory; $\gamma = 10$.

* It should be noted that for this figure and for all succeeding figures Poisson's ratio has been assigned the value of one-third. Also note that a split scale is used for the abscissas of these figures, thus allowing the results to be plotted for ζ between 0 and ∞.

support to the usual assumption made for problems of this type, namely that the bending stiffness of stringers can usually be ignored. Results will henceforth be given for the limiting cases of $B^2 = 0$ and $B^2 = \infty$.

Of primary interest in this analysis is the dependence of the solutions for $\bar{\tau}(\zeta)$ and $\bar{\sigma}(\zeta)$ on the shell curvature. The behavior of $\bar{\tau}(\zeta)$ is displayed in Fig. 6 for $B^2 = 0$ and $\gamma = 1$, 5 and 10. These results are contrasted with $\bar{\tau}(\zeta)$ as found for the Melan problem ($\gamma = 0$). Similarly, $\bar{\sigma}(\zeta)$ is displayed in Fig. 7 for the same values of B^2 and γ.

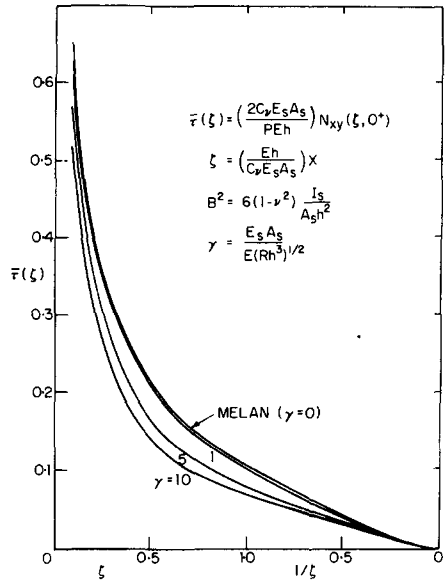

FIG. 6. $\bar{\tau}(\zeta)$ vs. ζ, shallow shell theory; $B^2 = 0$.

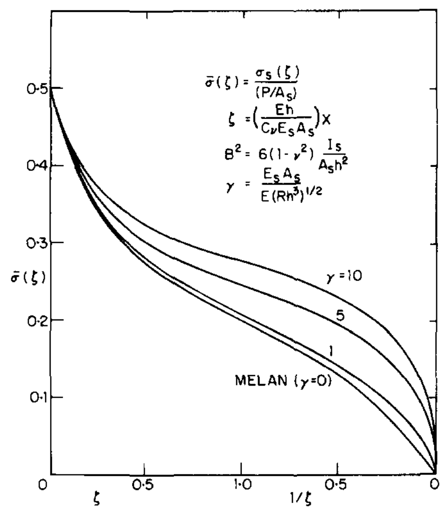

FIG. 7. $\bar{\sigma}(\zeta)$ vs. ζ, shallow shell theory; $B^2 = 0$.

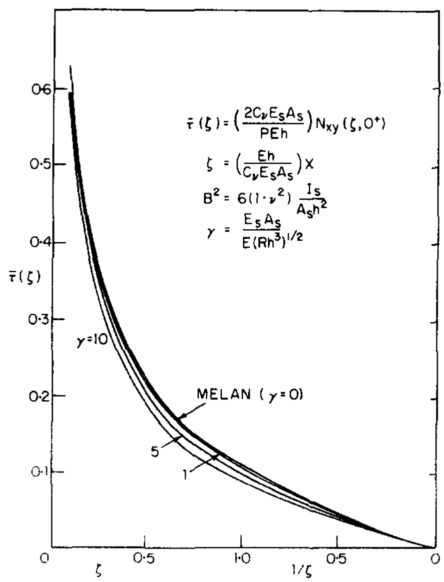

FIG. 8. $\bar{\tau}(\zeta)$ vs. ζ, shallow shell theory; $B^2 = \infty$.

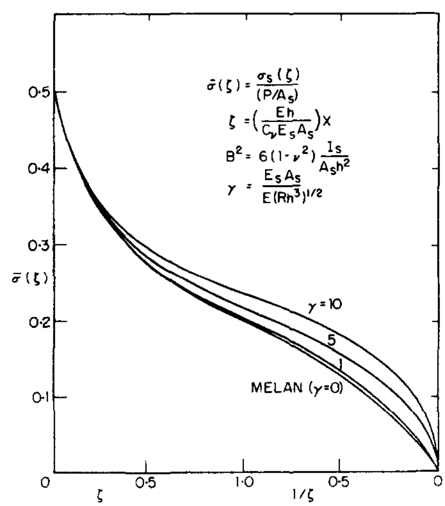

FIG. 9. $\bar{\sigma}(\zeta)$ vs. ζ, shallow shell theory; $B^2 = \infty$.

The behavior of $\bar{\tau}(\zeta)$ for $B^2 = \infty$ and $\gamma = 1, 5$ and 10 is given in Fig. 8. As in Fig. 6 these cylinder results are contrasted with the flat sheet Melan results ($\gamma = 0$). Similarly, $\bar{\sigma}(\zeta)$ is displayed in Fig. 9 for the same values of B^2 and γ. From a comparison of the results of Figs. 6 and 8, or 7 and 9 it is obvious that the solutions for $B^2 = \infty$ differ much less from the Melan results than do those for $B^2 = 0$ (for fixed γ). This is due to the fact that for $B^2 = \infty$, $w(x, y)|_{y=0} = 0$, which matches the condition along the stringer for the Melan problem ($w \equiv 0$). It should also be noted from either Fig. 7 or 9 that for a given value of ζ the stringer stress for the cylinder ($\gamma > 0$) is greater than that for the flat sheet ($\gamma = 0$). Due to the exponential decay of stress quantities in the circumferential direction, stress diffusion for the cylinder is not as efficient in this direction, and this probably accounts for the observed behavior of the axial stringer stress.

APPROACH BASED ON THE SIMMONDS' EQUATIONS FOR CIRCULAR CYLINDRICAL SHELLS

The complete circular cylindrical shell with an arbitrary number of equally spaced axial stringers will now be treated. The cases of one and two stringers are examined in some depth as they should offer the best basis on which to judge the performance and limitations of the shallow shell analysis. In particular one discrepancy already evident is that the shallow shell analysis predicts zero axial stringer stress at infinity (the ends of the stringers) due to the infinite parabolic shell cross section, whereas a finite non-zero stringer stress is anticipated on the basis of simple statics for the case of the finite circular shell cross section. Results are also given for the cases of six and ten stringers for the purpose of illustrating the effect of stringer spacing.

For the specific load system under study, i.e. a single concentrated force applied to and directed along each stringer, it is possible to reduce the domain of the boundary value problem to that portion of the shell bounded by a stringer and the generator halfway to an adjacent stringer. For the case of a single stringer, this domain will be half the shell.

On the basis of the conclusions drawn from the shallow shell analysis concerning the effects of bending stiffness of the stringer, only the enveloping solutions of zero and infinite bending stiffness will be sought. The specific boundary value problem illustrated in the following is that of zero bending stiffness, i.e. $B^2 = 0$. A shell theory for circular cylindrical shells due to Simmonds [7] will be used for this analysis. Starting from the linear theory of Sanders [11] and specializing to the case of circular cylindrical shells, Simmonds is able to obtain a particularly simple form for the equilibrium and compatibility equations through approximations introduced solely in the constitutive equations. These approximations introduce errors into the theory which have been shown to be negligible by Koiter's arguments [12].

Boundary value problem

The field equations for the Simmonds' theory expressed in terms of a stress potential function F, and the shell normal displacement component w, are

$$D(\nabla^4 w + w^{\cdot\cdot} + \lambda w'') + RF'' = 0 \tag{50}$$

$$(\nabla^4 F + F^{\cdot\cdot} + \lambda F'') - EhRw'' = 0 \tag{51*}$$

* Additional terms in this equation are present when surface loading exists.

where

$$
\begin{aligned}
(\ )' &= R(\ )_{,x} \\
(\ )^{\cdot} &= (\ )_{,\theta} = R(\ )_{,y} \\
\lambda &= \text{arbitrary, } O(1) \text{ constant.}
\end{aligned}
\tag{52}
$$

Equilibrium of the shell in the normal direction is expressed by (50) and (51) is the statement of compatibility of strains. For convenience, λ is equated to zero in what follows. Auxiliary equations for stress and displacement quantities in terms of F and w have also been developed and tabulated by Simmonds. Certain of these relations will be introduced as needed for this analysis. In general, stresses depend upon w as well as F for this theory.

The boundary conditions to be satisfied along the stringer at $y = 0$ and along the other bounding generator at $y = \pi R/N_s$, where N_s equals the number of axial stiffeners, will now be developed as expressions involving only F and w. The four physical conditions along the stringer $(-\infty < x < \infty, y = \theta = 0)$ are the same as encountered in the shallow shell analysis. The first of these is

$$v(x, y)|_{y=0} = u_\theta(x, \theta)|_{\theta=0} = 0 \tag{53}$$

which implies that $u_\theta''|_{\theta=0} = 0$. From Simmonds' auxiliary equations

$$u_\theta'' = -\left(\frac{1}{EhR}\right)[(2+\nu)F'' + F^{\cdot\cdot} + F]^{\cdot} \tag{54}$$

and hence b.c. (53) becomes

$$[(2+\nu)F'' + F^{\cdot\cdot} + F]^{\cdot}|_{\theta=0} = 0. \tag{55}$$

The next condition which also follows from symmetry is that the rotation of a shell element in the circumferential direction ϕ_θ, is zero along this line of stringer attachment. This condition

$$\phi_\theta|_{\theta=0} = -\frac{1}{R}(w^{\cdot} - u_\theta)|_{\theta=0} = 0 \tag{56}$$

together with (53) reduces to

$$w^{\cdot}|_{\theta=0} = 0. \tag{57}$$

Axial equilibrium of the stringer is expressed by

$$2\int_{-\infty}^{x} N_{\theta x}(\bar{x}, 0^+)\,d\bar{x} + [\sigma_s(x) - \sigma_s(-\infty)]A_s - P\,.\,H(x) = 0. \tag{58}$$

Treating the stringer as a rod and using the condition of axial strain continuity along this line of stringer attachment, one obtains

$$\sigma_s = E_s\varepsilon_s = E_s\varepsilon_x|_{\theta=0} = \frac{E_s}{R}u_x'|_{\theta=0}. \tag{59}$$

From Simmonds' auxiliary equations

$$u_x' = \frac{1}{EhR}(F^{\cdot\cdot} + F - \nu F''). \tag{60}$$

Introduction of (60) into (59) yields

$$\sigma_s = \frac{E_s}{EhR^2}(F^{\cdot\cdot}+F-\nu F'')|_{\theta=0}. \tag{61}$$

The following relation is also due to Simmonds:

$$N'_{\theta x} = -\frac{1}{R^2}\left\{F''-\left(\frac{D}{R}\right)[w^{\cdot\cdot}+w+(2-\nu)w'']\right\}^{\cdot}. \tag{62}$$

Equation (62) can be simplified by use of (57) to

$$N'_{\theta x}|_{\theta=0} = -\frac{1}{R^2}\left[F''-\left(\frac{D}{R}\right)w^{\cdot\cdot}\right]^{\cdot}|_{\theta=0}. \tag{63}$$

Differentiation of (58) with respect to x and the subsequent introduction of (61) and (63) into this equation yields

$$-\left(\frac{2}{R^2}\right)\left[F''-\left(\frac{D}{R}\right)w^{\cdot\cdot}\right]^{\cdot}+\frac{E_sA_s}{EhR^3}[F^{\cdot\cdot}+F-\nu F'']''|_{\theta=0} = P[\delta(x)]'. \tag{64}$$

The last condition concerns the resistance to bending offered by the stringer. For $B^2 = 0$ no transverse shear is exerted by the stringer along this shell edge. The relation for this transverse shear from Simmonds' auxiliary equations is

$$R_\theta = -\frac{D}{R^3}[w^{\cdot\cdot}+w+(2-\nu)w'']^{\cdot}. \tag{65}$$

Use of (65) together with (57) yields

$$w^{\cdot\cdot\cdot}|_{\theta=0} = 0 \tag{66}$$

for this boundary condition. Use of (66) in (64) yields

$$-\left(\frac{2}{R^2}\right)F''^{\cdot}+\frac{E_sA_s}{EhR^3}[F^{\cdot\cdot}+F-\nu F'']''|_{\theta=0} = P[\delta(x)]'. \tag{67}$$

The boundary conditions along the other bounding generator ($-\infty < x < \infty$, $\theta = \bar{\theta} = \pi/N_s$) are similarly obtained from symmetry considerations and can be written as

$$w^{\cdot}|_{\theta=\bar{\theta}} = w^{\cdot\cdot\cdot}|_{\theta=\bar{\theta}} = F^{\cdot}|_{\theta=\bar{\theta}} = F^{\cdot\cdot\cdot}|_{\theta=\bar{\theta}} = 0. \tag{68}$$

The field equations (50) and (51) subject to the boundary conditions (55), (57) and (66) through (68), together with the requirement of boundedness as $|x| \to \infty$, constitute the boundary value problem for the infinite strip domain ($-\infty < x < \infty$, $0 < y < \pi R/N_s$) to be treated. As before these equations will be cast into a more concise non-dimensional form. Through the use of (16) the field equations (50) and (51) become

$$\bar{\nabla}^4\bar{w}+\bar{A}^2\bar{w}_{,\eta\eta}+\beta^4\bar{F}_{,\xi\xi} = 0 \tag{69}$$

$$\bar{\nabla}^4\bar{F}+\bar{A}^2\bar{F}_{,\eta\eta}-\bar{w}_{,\xi\xi} = 0 \tag{70}$$

respectively. The parameter $\bar{A} = 2\pi E_sA_s/(EA_c)$ is essentially the ratio of the stringer stretching stiffness to that of the circular shell and is completely determined by specification of γ and the shell thickness ratio (h/R), i.e. $\bar{A} = (h/R)^{\frac{1}{2}}\gamma$. The four b.c. along the stringer, (55), (57), (66) and (67), become

$$\begin{aligned} \bar{F}_{,\eta\eta\eta}+(2+\nu)\bar{F}_{,\xi\xi\eta}+\bar{A}^2\bar{F}_{,\eta}|_{\eta=0} &= 0 \\ \bar{w}_{,\eta}|_{\eta=0} &= 0 \\ \bar{w}_{,\eta\eta\eta}|_{\eta=0} &= 0 \\ -2\bar{F}_{,\xi\eta}+\bar{F}_{,\xi\eta\eta}-\nu\bar{F}_{,\xi\xi\xi}+\bar{A}^2\bar{F}_{,\xi}|_{\eta=0} &= \delta(\xi) \end{aligned} \tag{71}$$

respectively, and the four conditions along the generator at $\bar{\eta} = \pi/N_s\bar{A}$, Equations (68), become respectively,

$$\bar{w}_{,\eta}|_{\eta=\bar{\eta}} = \bar{w}_{,\eta\eta\eta}|_{\eta=\bar{\eta}} = \bar{F}_{,\eta}|_{\eta=\bar{\eta}} = \bar{F}_{,\eta\eta\eta}|_{\eta=\bar{\eta}} = 0. \tag{72}$$

At this point the two differences arising in the boundary value problem formulation for $B^2 = \infty$ will be cited:

(1) the third b.c. of (71) becomes

$$\bar{w}|_{\eta=0} = 0 \tag{73}$$

(2) the last b.c. of (71) becomes

$$\begin{aligned} -2\bar{F}_{,\xi\xi\eta}+\frac{\bar{A}^2}{6(1-\nu^2)\gamma^4}\bar{w}_{,\eta\eta\eta}+\bar{F}_{,\xi\xi\eta\eta} \\ -\nu\bar{F}_{,\xi\xi\xi\xi}+\bar{A}^2\bar{F}_{,\xi\xi}|_{\eta=0} = [\delta(\xi)]_{,\xi}. \end{aligned} \tag{74}$$

Note that as $\bar{A}^2 \to 0$, corresponding to $(A_s/A_c) \to 0$, the above field equations with associated boundary conditions along the line of attachment of the stringer reduce to the corresponding relations for shallow shell theory and that the other boundary goes off to infinity. Hence, it appears that the shallow shell solution is the asymptotic solution to the complete shell problem in the limit as $\bar{A} \to 0$, holding γ fixed.

Solution and results

Fourier transforms will again be used to effect the solution of the boundary value problem as stated in the previous section. Upon transformation with respect to ξ, the field equations (69) and (70) become

$$\tilde{w}_{,\eta\eta\eta\eta}-2\omega^2\tilde{w}_{,\eta\eta}+\omega^4\tilde{w}+\bar{A}^2\tilde{w}_{,\eta\eta}-\beta^4\omega^2\tilde{F} = 0 \tag{75}$$

$$\tilde{F}_{,\eta\eta\eta\eta}-2\omega^2\tilde{F}_{,\eta\eta}+\omega^4\tilde{F}+\bar{A}^2\tilde{F}_{,\eta\eta}+\omega^2\tilde{w} = 0. \tag{76}$$

Exponential solutions of the form $\exp(\alpha\eta)$ are appropriate and their use in (75) and (76) yields the characteristic equation for the exponent coefficients $\alpha(\omega)$

$$[(\alpha^2-\omega^2)^2+\bar{A}^2\alpha^2]^2+\beta^4\omega^4 = 0. \tag{77}$$

The roots of (77) for α^2 are

$$\begin{aligned} \alpha_1{}^2 &= \tfrac{1}{2}[H+z_1{}^{\frac{1}{2}}] \\ \alpha_2{}^2 &= \tfrac{1}{2}[H-z_1{}^{\frac{1}{2}}] \\ \alpha_3{}^2 &= \tfrac{1}{2}[H+z_1^{*\frac{1}{2}}] = \alpha_1^{*2} \\ \alpha_4{}^2 &= \tfrac{1}{2}[H-z_1^{*\frac{1}{2}}] = \alpha_2^{*2} \end{aligned} \tag{78}$$

where

$$H = 2\omega^2 - \bar{A}^2 \tag{79}$$

$$z_1 = [\bar{A}^4 - 4\bar{A}^2\omega^2 + i4\beta^2\omega^2] \tag{80}$$

and the square root convention is the same as before. The eight roots of (77) for α may be written concisely in the notation of (78) as

$$\begin{aligned} \alpha_1 &= -({\alpha_1}^2)^{\frac{1}{2}} = -\alpha_5 \\ \alpha_2 &= -({\alpha_2}^2)^{\frac{1}{2}} = -\alpha_6 \\ \alpha_3 &= -({\alpha_3}^2)^{\frac{1}{2}} = -\alpha_7 = \alpha_1^* \\ \alpha_4 &= -({\alpha_4}^2)^{\frac{1}{2}} = -\alpha_8 = \alpha_2^*. \end{aligned} \tag{81}$$

The complete solutions with as yet undetermined coefficients are

$$\begin{aligned} \tilde{F}(\omega, \eta) &= \sum_{n=1}^{8} A_n(\omega)\exp(\alpha_n\eta) \\ \tilde{w}(\omega, \eta) &= \sum_{n=1}^{8} B_n(\omega)\exp(\alpha_n\eta) \end{aligned} \tag{82}$$

where A_n and B_n are not independent quantities but are related according to the relation

$$B_n = p_n A_n \tag{83}$$

where

$$p_1 = p_2 = -p_3 = -p_4 = p_5 = p_6 = -p_7 = -p_8 = -i\beta^2. \tag{84}$$

As before, attention will be directed to $\tilde{N}_{yx}$ since knowledge of this function is sufficient for the determination of both the transmitted shearing stress and the stringer stress. For the case of $B^2 = 0$,

$$\tilde{N}_{yx}|_{\eta=0} = i\omega\tilde{F}_{,\eta}|_{\eta=0} = i\omega\sum_{n=1}^{8} A_n(\omega)\alpha_n(\omega) \tag{85}$$

which is analogous to the expression encountered in the shallow shell analysis. For $B^2 = \infty$ the expression giving $\tilde{N}_{yx}$ is somewhat more complicated due to the additional dependence upon $\tilde{w}$. This expression is

$$\begin{aligned} -i\omega\tilde{N}_{yx}|_{\eta=0} &= \omega^2\tilde{F}_{,\eta} + \lambda_0 w_{,\eta\eta\eta}|_{\eta=0} \\ &= \sum_{n=1}^{8} [\omega^2 A_n\alpha_n + \lambda_0 B_n{\alpha_n}^3] \end{aligned} \tag{86}$$

where $\lambda_0 = \bar{A}^2/12(1-\nu^2)\gamma^4$.

The coefficients $A_n(\omega)$, and hence $B_n(\omega)$, or more importantly the particular combinations of these quantities as required in (85) and (86) are determined by application of the b.c. (71) and (72). The appropriate form for these b.c. is obtained by application of the

Fourier transform as before. For $\eta = 0$,

$$\begin{aligned} \tilde{F}_{,\eta\eta\eta} - (2+\nu)\omega^2 \tilde{F}_{,\eta} + \bar{A}^2 \tilde{F}_{,\eta} &= 0 \\ \tilde{w}_{,\eta} &= 0 \\ \tilde{w}_{,\eta\eta\eta} &= 0 \\ -2\tilde{F}_{,\eta} + \tilde{F}_{,\eta\eta} + \nu\omega^2 \tilde{F} + \bar{A}^2 \tilde{F} &= i/\omega \end{aligned} \tag{87}$$

and for $\eta = \bar{\eta} = \pi/(N_s \bar{A})$,

$$\tilde{w}_{,\eta} = \tilde{w}_{,\eta\eta\eta} = \tilde{F}_{,\eta} = \tilde{F}_{,\eta\eta\eta} = 0 \tag{88}$$

become the appropriate expressions.

When $B^2 = 0$, (85) is pertinent and

$$\tilde{N}_{yx} = \mathrm{Re}[C] \tag{89}$$

where,

$$C = (c-b)\,.\,[(f\,.\,s - l\,.\,q)(c+d)/(s-q) + (m\,.\,r - g\,.\,t)(b+d)/(t-r)]^{-1} \tag{90}$$

and

$$\begin{aligned} b &= \alpha_1{}^2 \\ c &= \alpha_2{}^2 \\ d &= [\bar{A}^2 - (2+\nu)\omega^2] \\ e &= -(\bar{A}^2 + \nu\omega^2) \\ f &= e/\alpha_1 + 2 - \alpha_1 = -l + 4 \\ g &= e/\alpha_2 + 2 - \alpha_2 = -m + 4 \\ \bar{\eta} &= \pi/(N_s \bar{A}) \\ q &= \exp(\alpha_1 \bar{\eta}) = 1/s \\ r &= \exp(\alpha_2 \bar{\eta}) = 1/t. \end{aligned} \tag{91}$$

When $B^2 = \infty$, the pertinent expression is (86) and

$$\tilde{N}_{yx}|_{\eta=0} = \frac{(\omega^2 P_2 - \lambda_0 \beta^2 P_1)}{(P_1\,.\,P_4 + P_2\,.\,P_3)} \tag{92}$$

where

$$\begin{aligned} P_1 &= \mathrm{Im}\{[s(c+d)(1+q^2)/(1-q^2) \\ &\quad - t(b+d)(1+r^2)/(1-r^2)]/[c-b]\} \\ P_2 &= \mathrm{Re}\{[t(1+r^2)/(1-r^2) - s(1+q^2)/(1-q^2)]/[c-b]\} \\ P_3 &= \mathrm{Re}\{(c+d)(f - l\,.\,q^2)/(1-q^2) \\ &\quad - (b+d)(g - m\,.\,r^2)/(1-r^2)]/[c-b]\} \\ P_4 &= \mathrm{Im}\{[(g - m\,.\,r^2)/(1-r^2) - (f - l\,.\,q^2)/(1-q^2)]/[c-b]\} \end{aligned} \tag{93}$$

and

$$
\begin{aligned}
b &= \alpha_1{}^2 \\
c &= \alpha_2{}^2 \\
d &= [\bar{A}^2-(2+\nu)\omega^2] \\
e &= -(\bar{A}^2+\nu\omega^2)\omega^2 \\
f &= (e+2\omega^2\alpha_1-\omega^2\alpha_1{}^2-i2\lambda_0\beta^2\alpha_1{}^3]/\alpha_1 \\
g &= (e+2\omega^2\alpha_2-\omega^2\alpha_2{}^2-i2\lambda_0\beta^2\alpha_2{}^3]/\alpha_2 \\
l &= -(e-2\omega^2\alpha_1-\omega^2\alpha_1{}^2+i2\lambda_0\beta^2\alpha_1{}^3]/\alpha_1 \\
m &= -(e-2\omega^2\alpha_2-\omega^2\alpha_2{}^2+i2\lambda_0\beta^2\alpha_2{}^3]/\alpha_2 \\
q &= \exp(\alpha_1\bar{\eta}) \\
r &= \exp(\alpha_2\bar{\eta}) \\
s &= -1/\alpha_1 \\
t &= -1/\alpha_2 .
\end{aligned}
\tag{94}
$$

Once again, as in the shallow shell analysis, integral representations can be obtained for $\bar{\tau}(\zeta)$ and $\bar{\sigma}(\zeta)$ which can be evaluated by the process of numerical quadrature discussed earlier. For the same values of γ considered previously, i.e. $\gamma = 1, 5, 10$, and for various numbers of stringers the resultant shear stress and stringer normal stress were obtained. The parameter $\bar{A}$ must also be specified for these calculations. It is helpful to note that

$$\bar{A} = \gamma\,.\,(h/R)^{\frac{1}{2}}. \tag{95}$$

As has been previously mentioned, the shallow shell analysis is recovered from this analysis in the limit as $\bar{A} \to 0$ holding γ fixed, i.e. $(h/R) \to 0$. A critical test for the shallow shell results is obtained by specifying (h/R) to be 0·01 in the present analysis.

The calculated values for $\bar{\tau}(\zeta)$ when $N_s = 1$ or 2 were found not to differ appreciably from those values found from the one-stringer shallow shell analysis for this function. Hence, Figs. 6 and 8 represent quite adequately the behavior of $\bar{\tau}(\zeta, \gamma)$ for $B^2 = 0$ and $B^2 = \infty$ respectively, as found from this "exact" analysis. The stringer stress is a more sensitive measure of the stress diffusion for this problem since it reflects the cumulative effect of the transmitted shear stress. That is, although a significant deviation between curves for $\bar{\tau}(\zeta)$ might not be present, the accumulation of this deviation may become quite important.

A specific comparison between the shallow shell results and the results of the present analysis is given by Figs. 10 and 11 for $N_s = 1$ and $N_s = 2$ respectively. For these figures $B^2 = 0$ and $\gamma = 1$, 5 and 10. The results for $B^2 = \infty$ display a completely analogous behavior. The better agreement of results for $N_s = 2$ than for $N_s = 1$ may at first seem strange due to the relative stringer spacing, but it is completely acceptable on the basis that $\bar{\sigma}(\infty)$ is greater for $N_s = 1$ due to the overall moment at infinity which exists for this case. For $N_s \geq 2$, no such moment exists and increasing N_s brings about increasing differences between the results of this analysis and the single-stringer shallow shell results as illustrated by Figs. 12 and 13 for $B^2 = 0$ and $B^2 = \infty$, respectively. This result was to be

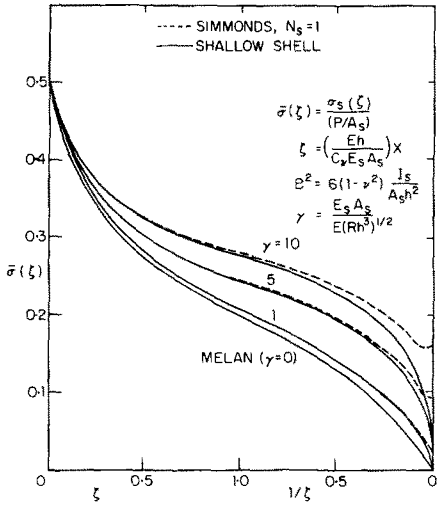

FIG. 10. $\bar{\sigma}(\zeta)$ vs. ζ, comparison of results for shallow shell and Simmonds' theories: $B^2 = 0$, $N_s = 1$, $R/h = 100$.

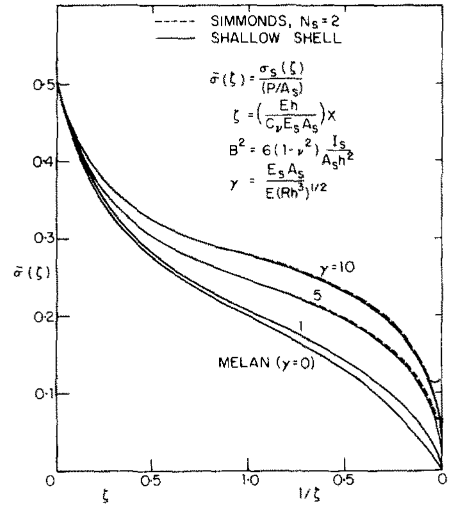

FIG. 11. $\bar{\sigma}(\zeta)$ vs. ζ, comparison of results for shallow shell and Simmonds' theories: $B^2 = 0$, $N_s = 2$, $R/h = 100$.

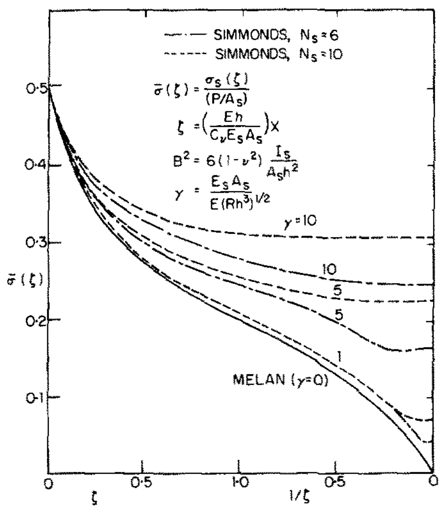

FIG. 12. $\bar{\sigma}(\zeta)$ vs. ζ, Simmonds' theory: $B^2 = 0$, $R/h = 100$.

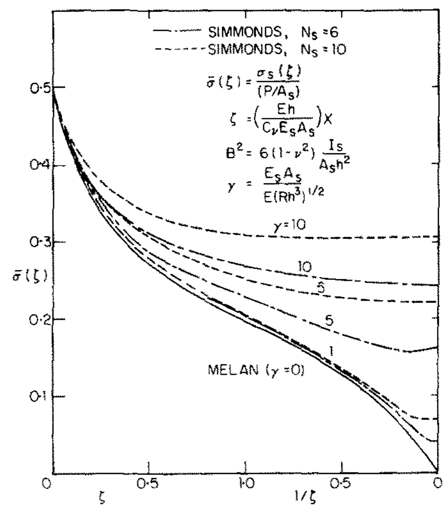

FIG. 13. $\bar{\sigma}(\zeta)$ vs. ζ, Simmonds' theory: $B^2 = \infty$, $R/h = 100$.

anticipated since the single-stringer shallow shell analysis included here is based on the assumption that stringers are spaced sufficiently far apart so that any effects due to the presence of adjacent stringers can be neglected. Comparison of Figs. 12 and 13 shows that as the number of stringers is increased for shells of moderately large curvature, thus increasing the stretching stiffness of the composite structure, the effect of stringer bending stiffness becomes greatly diminished.

CONCLUDING REMARKS

In conclusion it should be noted that significant differences arise with regard to the transmitted shearing stress and stringer normal stress of Melan's load diffusion problem when curvature transverse to the stringer is introduced. The singularity of the shear function at the point of load application is unaltered, as was to be expected on physical grounds, but a finite modification there and over the entire stringer length is necessitated. The membrane shear stress transmitted to the cylinder is less than that for the flat sheet over nearly the entire length, and the stringer normal stress is consequently greater.

For the situation of complete circular cylindrical shells, the one-stringer shallow shell approach yields quite good results for the stress quantities of interest provided the stringers are not too closely spaced and provided these results are not used in the immediate vicinity of the stringer ends. The assumptions made in developing the shallow shell results also admit the possibility that these results might pertain to cylindrical shells other than circular, e.g. elliptical, if some discretion is used in their application.

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(*Received* 1 *May* 1968)

Абстракт—Решается задача диффузии нагрузки для цилиндрических упругих оболочек. Исследуются бесконечно длинные оболочки круглого поперечного сечения и постоянной толщины. Зти цилиндры укреплены одним или более равномерно расположенными и непрерывно прикрепленными осевыми ребрами жесткости. Каждое ребро жесткости загруженное одной концентрической продольной силой. Величина, направление и осевое положение каждой силы одинаковы. Величины напряжений в ироцентах являютсн мембранными сдвигающими напряжениями, которые передаются через нагруженное ребро жесткости на оболочку, и осевыми напряжениями, возникающими внутри ребра. Иллюстрируется порядок зтих величин напряжений для каждой кривизны оболочки, расстояния ребер жесткости и зффекта жесткости при изгибе ребер.